Relations on sets: in principle

1. If A is a set, what do we mean by a [binary] ***relation*** on the set A?
   (a) Conceptually, what *logical* concept we've already encountered is the same thing as a relation?
   (b) Give some examples of relations...
      (i) on the set $\mathbb{Z}$
      (ii) on the set of vertices in a graph G
      (iii) on the set of subgraphs of a graph G

2. Properties of relations:
   (a) What does it mean for a relation to be ***reflexive***?
   Give examples of reflexive and non-reflexive relations from those you listed above.
   (b) What does it mean for a relation to be ***symmetric***?
   Give examples of symmetric and non-symmetric relations.
   (c) What does it mean for a relation to be ***transitive***?
   Give examples of transitive and non-transitive relations.

3. Equivalence relations:
   (a) What is an ***equivalence relation*** on a set A?
   (b) How does an A give us an ***equivalence class*** (often written $[a]$, $\overline{a}$, or $\mathbf{a}$) for each element $a \in A$?
   (c) What properties make these equivalence classes a ***partition*** of the set A?
   How do we typically denote the collection of *all* equivalence classes of A under some relation $\sim$?
   (d) How does *any* relation R "generate" an equivalence relation, both intuitively and formally?

Integer relations: divisibility, congruence, and modular arithmetic

4. Define a relation $\mid$ on $\mathbb{Z}$ via: $a \mid b$ (read "a divides b") just when $\underline{\exists\, k \in \mathbb{Z} \text{ such that } b = ka}$.
   (a) Prove that $4 \mid 12$, $6 \mid 12$, and $12 \mid 12$, but $24 \nmid 12$ and $5 \nmid 12$.
   (b) Which of the above properties does this relation possess?

5. Fix a positive integer n.
   We say that $a, b \in \mathbb{Z}$ are "congruent modulo n", written $a \equiv b \pmod{n}$, just when $\underline{n \mid (b - a)}$.
   (a) Prove that $6 \equiv 1 \pmod{5}$ but that $4 \not\equiv 1 \pmod{5}$.
   (b) Prove that congruence modulo n is an equivalence relation on $\mathbb{Z}$.
   What are its equivalence classes, for $n = 1$, $n = 2$, and $n = 5$?
   In general, what are this relation's equivalence classes?

6. Again fix a positive integer n. Prove that addition, subtraction, and multiplication respect equivalence classes modulo n (i.e., equivalent operands give equivalent results) as below, giving us ***modular arithmetic***).
   Specifically, supposing that $a, a', b, b' \in \mathbb{Z}$ satisfy $a \equiv a'$ and $b \equiv b' \pmod{n}$, prove that:
   (a) $a + b \equiv a' + b' \pmod{n}$;
   (b) $a - b \equiv a' - b' \pmod{n}$; and
   (c) $a \cdot b \equiv a' \cdot b' \pmod{n}$;

   These properties allow us to think of performing arithmetic operations on the set of *equivalence classes* $\left\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\right\}$ mod n—e.g., mod 6, we have $\overline{2} \cdot \overline{3} = \overline{6} = \overline{0}$.

Some relations on graphs

7. Suppose that G is a simple [undirected] graph. Consider the relations given below on the vertices of G.
   (a) $v_0\ A\ v_1$ if there is an edge joining the vertices v_0 and v_1. (What word might the "A" stand for?)
   (b) $v_0\ C\ v_1$ if there is a path joining the vertices v_0 and v_1. (What word might the "C" stand for?)

   Which of these two relations is an equivalence relation? What are its equivalence classes? How does the other relation relate to it?

8. Suppose that G is a digraph, and define a relation on the vertices of G by $v_0\ SC\ v_1$ just when there are *directed* paths both from v_0 to v_1 *and* from v_1 to v_0 (such vertices are called ***strongly connected***).
   (a) Show that SC is an equivalence relation on the vertices of G (the figure below gives you a digraph to think about).
   (b) Circle the equivalence classes for SC for the digraph below.
   (c) If we make a new graph $\overline{G}$ with a vertex for each *equivalence class* of SC and add all edges of G that connect vertices from different equivalence classes, explain why this new directed graph must be a DAG (directed acyclic graph).

   [Hint: what would a cycle on this new graph mean for G and the equivalence classes?]